A Modern Approach to an Ancient Quantity

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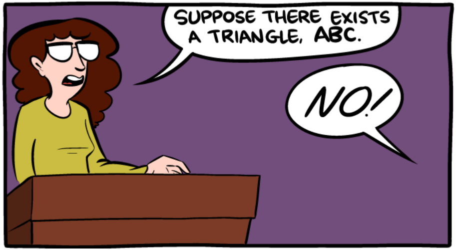

What is Geometry?

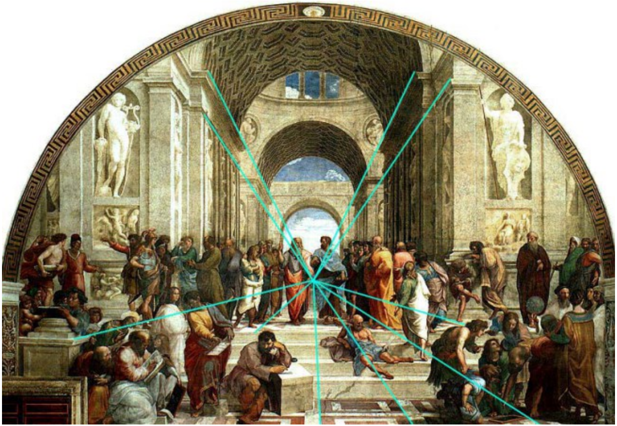

Figure 1: La Escuela de Atenas de Raphael; Un ejemplo de geometría en perspectiva.

Invariants of Groups of Transformations

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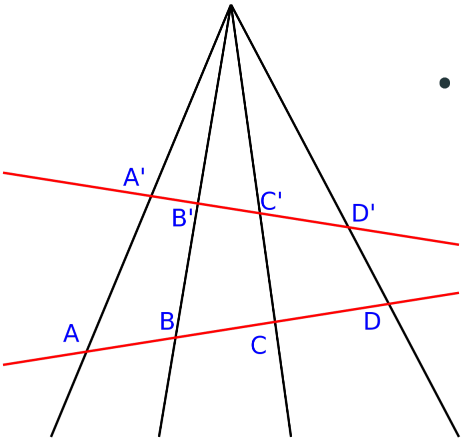
A Ratio of Ratios



- Definition of the cross ratio:

$$(A, B, C, D) = \frac{AC/AD}{BC/BD}.$$

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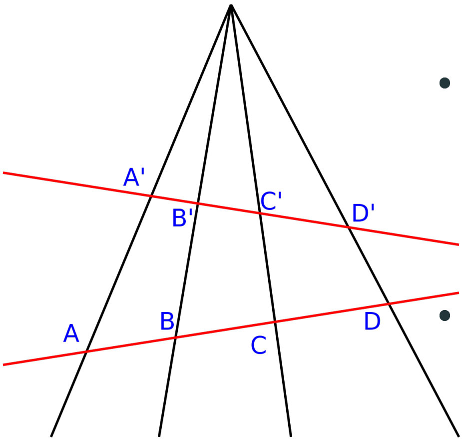

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$$(A, B, C, D) = \frac{AC/AD}{BC/BD}.$$

- It turns out

$$(A, B, C, D) = (A', B', C', D').$$

The Classical Cross Ratio

Definition
Given four finite distinct points $A = z_1$, $B = z_2$, $C = z_3$, and $D = z_4$ in the complex plane, the cross ratio is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{AC/AD}{BC/BD}.$$

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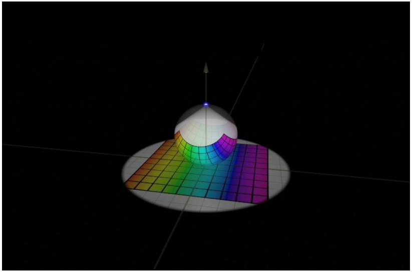
What happens if one of the points is 'the point at infinity'?

What happens if we permit, say, $z_1 = z$ to be a variable?

A linear fractional map is defined as

$$\phi(z) = \frac{az + b}{cz + d}$$

where a, b, c, and d are complex numbers and $ad - bc \neq 0$.



The Cross Ratio as a LFM

Theorem
Cross ratios are invariant under LFMs. For a LFM ϕ such that $\phi(z_i) = w_i$, for distinct z_i's and w_i's, we have

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)}$$

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There is a unique LFM that can interpolate two sets of three distinct points!

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$$m_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Projective Coordinates

Given $v = (v_1, v_2) \in \mathbb{C}^2$ where $v_1 \in \mathbb{C}$ and $v_2 \in \mathbb{C}$ with $v \neq (0,0)$, we identify $v \sim \frac{v_1}{v_2} \in \overline{\mathbb{C}}$.

What's Really Going On

A linear transformation in $\mathbb{C}^2$ can be represented by a complex 2×2 matrix as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}.$$

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Let $z \sim \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $w \sim \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then we have

$$w = \frac{w_1}{w_2} = \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{a\left(\frac{z_1}{z_2}\right) + b}{c\left(\frac{z_1}{z_2}\right) + d} = \frac{az + b}{cz + d}.$$

The Cross Ratio in Projective Coordinates

Definition

If $z = (z_1, z_2)$ and $w = (w_1, w_2)$ are points in $\mathbf{CP}^1$, then we define

$$[z, w] = \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}.$$

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Let w_1, w_2, w_3, and w_4 be four distinct points in $\mathbf{CP}^1$, then we define the cross ratio to be

$$(w_1, w_2, w_3, w_4) = \frac{[w_1, w_3][w_2, w_4]}{[w_1, w_4][w_2, w_3]}$$

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Really!

Linear Fractional Maps in $\mathbb{C}^N$.

Definition
A map ϕ is called a linear fractional map if

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, and $D \in \mathbb{C}$.

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What might m_ϕ look like now?

Example

Let ϕ be the linear fractional map in two complex variables given by

$$\phi(z) = \phi(z_1, z_2) = \left(\frac{z_1 + 1}{-z_1 + 3}, \frac{2z_2}{-z_1 + 3} \right).$$

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Identifying $\langle z, C \rangle$ with $C^* z$, we can write this as

$$\left(\frac{z_1 + 1}{-z_1 + 3}, \frac{2z_2}{-z_1 + 3} \right) = \frac{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(-1, 0)^T (z_1, z_2) + 3}.$$

with $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = (1, 0)^T$, $C = (-1, 0)^T$, and $D = 3$.

Definition (Π)
Given five distinct points w_i for $i = 1, ..., 5$ in $\mathbf{CP}^2$, we define the cross ratio as

$$(w_1, w_2, w_3, w_4, w_5) = \frac{[w_1, w_3, w_5][w_2, w_4, w_5]}{[w_1, w_4, w_5][w_2, w_3, w_5]} = \frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}.$$

Generalizing to Two Variables

Definition (Π)
We define the cross ratio pair in $\mathbf{CP}^2$ as

$$(z_1, z_2, z_3, z_4, z_5)_2 = \left(\frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}, \frac{[z_1, z_3, z_4][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_4]} \right)$$

where we see that this defines a linear fractional map when the point associated with z_1 is a variable in $\mathbb{C}^2$.